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SELECTION OF PRIOR DISTRIBUTIONS FOR BAYESIAN RELIABILITY ASSESSMENT

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Kelly Air Force Base, Texas

May 1973

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SELECTION OF PRIOR DISTRIBUTIONS FOR BAYESIAN RELIABIBITY ASSESSMENT

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(See reverse)

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20. Abstract (cont'd).

matching technique is used. There are an infinite number of choices, but it can be proved that component prior distributions can always be chosen to fall within a certain range, and still give the desired system outcome provided the desired system prior distribution is well-behaved. Results are obtained by considering the case for a system composed of two independent components in series, a system composed of two independent components in parallel, and the more general system composed of a combination of the preceding two. One has the following three theorems:

<u>THEOREM I</u>: If S^*, F^* are independent series system prior parameters such that $-1<S^*,F^*<2$ and $-2<T^*<1$ then we may find component prior paramenters S,F, such that $-1<F<1$, $-1<S<2$ and $-2<T<1$ which give S^* and F^* for system paramenters. If in addition $F^*<1$ then $-1<F<0$.

<u>THEOREM II</u>: If S^*, F^* are independent parallel system prior paramenters such that $-1<S^*,F^*<2$ and $-2<T^*<1$ then we may find component prior parameters S,F such that $-1<S<1$, $-1<F<2$ and $-2<T<1$ which give S^* and F^* for system parameters. If in addition $S^*<1$ then $-1<S<0$.

<u>THEOREM III:</u> If the desired system prior paramenters S^*,F^* satisfy $-1<S^*,F^*<2$ and $-2<T^*<1$ then there exist a set of component prior parameters S,F which fall into the same limits and give the system the desired prior distribution.

ABSTRACT

A necessary part of the Bayesian method is the choice of the prior distribution. In reliability assessment it is often desirable to obtain prior distributions for the components of a system which give a uniform prior distribution for the whole system when a moment-matching technique is used. There are an infinite number of choices, but it can be proved that component prior distributions can always be chosen to fall within a certain range, and still give the desired system outcome provided the desired system prior distribution is well behaved. Results are obtained by considering the case for a system composed of two independent components in series, a system composed of two independent components in parallel, and the more general system composed of a combination of the preceding two. One has the following three theorems:

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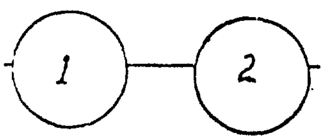
SELECTION OF PRIOR DISTRIBUTIONS FOR BAYESIAN RELIABILITY ASSESSMENT

A unique problem which confronts the user of Bayesian methods is how to choose the proper prior distributions. In applications to reliability assess - ment of mechanical/electrical systems[1] this amounts to choosing prior distributions at the component level which give a uniform($f(R)=1$) prior at the system level. The question that naturally arises is "What kind of control do we have over these priors, i.e. between what limits can we trap them?" To avoid prior distributions dom- inating the test data it is desirable to choose beta curves[2] with test parameters as small as possible. It is the purpose of this paper to show precisely how well this can be done.

The user of these methods is quite aware of the fact that some of the parameters for beta curves invari- ably turn out to be negative, i.e. for most systems a beta curve prior with equation $B(S,F)R^{S}(1-R)^{F}$, $B(S,F)$ the beta constant, will have one of S,F negative. This is an unfortunate situation because priors are supposed to reflect prior knowledge, and $S<0$ implies that the com- ponent has failed more than it has been tested, while $F<0$ implies that the component has worked more than it has been tested, both of which are impossible. These situations have been encountered where for simplicity all components in a series have been assigned the same prior, and the same for components in parallel. One might hope to get around this by allowing different priors for different components. For example, for a system composed of two components in series,



, one might try to give (1) and (2)

different priors to make the system uniform, and thus eliminate any negative parameters. If parameters for (i) are S_i, T_i then the system mean is

$$\frac{S_1+1}{T_1+2}\,\frac{S_2+1}{T_2+2} = .5 \quad \text{(mean of a uniform distribution)}$$

and second moment

$$\frac{S_1+1}{T_1+2}\,\frac{S_1+2}{T_1+3}\,\frac{S_2+1}{T_2+2}\,\frac{S_2+2}{T_2+3} = \frac{1}{3} \quad \text{(second moment of a unif.dis.)}$$

Thus $$\frac{S_1+1}{T_1+2}\,\frac{S_2+1}{T_2+2} = .5$$

$$\frac{S_1+2}{T_1+3}\,\frac{S_2+2}{T_2+3} = \frac{2}{3}$$

So we get

(a) $2S_1S_2 + 2S_1 + 2S_2 + 2 = T_1T_2 + 2T_1 + 2T_2 + 4$

(b) $3S_1S_2 + 6S_1 + 6S_2 + 12 = 2T_1T_2 + 6T_1 + 6T_2 + 18$

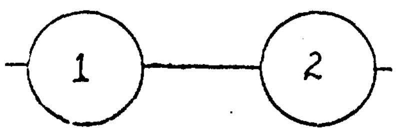
3(a)-(b) implies $3S_1S_2 = T_1T_2$

Therefore (a) gives $2(T_1-S_1)+2(T_2-S_2) = -S_1S_2-2$

If both $F_1, F_2 > 0$ then $-S_1S_2-2 > 0$ or $-2 > +S_1S_2$ so one of the S_i's is negative.

Hence any hope of conquering the problem this way is futile. Thus it does no real harm to assign the same priors to, for example, both (1) and (2), and it simplifies the calculation enormously. Since all systems can be broken down successively into either two things in series or two things in parallel, it will suffice to prove theorems for those two cases.

INDEPENDENT SERIES SYSTEM



The uniform system prior is just one of many system priors. The theorems will be proved for the more general case of a desired system prior with $-2 < T^* < 1$, $-1 < S^* < 2$, $-1 < F^* < 2$. The uniform case has $S^* = F^* = T^* = 0$. Suppose then that we desire a system prior which falls within that range. The system mean m and second moment mm will come from parameters in the above range. If S and T are the component parameters (same ones being assigned to both (1) and(2)), then

$$m = \frac{(S+1)^2}{(T+2)^2}$$

$$mm = \frac{m(S+2)^2}{(T+3)^2}$$

Solving we get

$$T = \frac{1+2\sqrt{m} - 3\sqrt{mm/m}}{\sqrt{mm/m} - \sqrt{m}}$$

$$S = \frac{2-\sqrt{mm}/m - \sqrt{mm/m}}{\sqrt{mm}/m - 1}$$

$$F = T-S$$

<u>THEOREM A:</u> $-1 < F < 1$

Proof: $F > -1$ iff $\dfrac{1-2\sqrt{mm/m} + \sqrt{mm}}{\sqrt{mm/m} - \sqrt{m}} > -1$

iff $1-\sqrt{mm/m} + \sqrt{mm} - \sqrt{m} > 0$

iff $1-\sqrt{m} > \sqrt{mm}(1/\sqrt{m} - 1)$

iff $1-\sqrt{m} > \sqrt{mm}((1-\sqrt{m})/\sqrt{m})$

iff $1 > \sqrt{mm/m}$, true since

$mm/m = \frac{S^*+2}{T^*+3} < 1$ since $F^* > -1$ where

S^*, F^*, T^* are the desired system parameters.

$F<1$ if $1+2\sqrt{m} \; -3\sqrt{mm/m} \; -2\sqrt{m} + \sqrt{mm/m} + \sqrt{mm} - \sqrt{mm/m} + \sqrt{m} < 0$

if $1-3\sqrt{mm/m} + \sqrt{m} + \sqrt{mm} < 0$

if $f(x,y) = 1-3\sqrt{y} + \sqrt{x} + \sqrt{xy} < 0$

where $x = \frac{S^*+1}{T^*+2}$, $y = \frac{S^*+2}{T^*+3}$

Since this is a proof which uses a method common to several proofs, let me elaborate a little. Solving for S^*, T^* we get

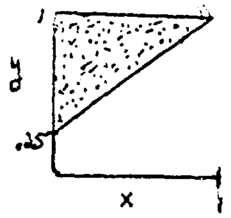
$$T^* = \frac{1+2x-3y}{y-x} \qquad S^* = \frac{-y+2x-xy}{y-x}$$

$S^* > -1$ implies $x > 0$

$F^* > -1$ implies $x, y < 1$

$T^* < 1$ implies $y > .25 + .75x$

So the region R that x,y fall into is



It suffices to show $f \leq 0$ on this region.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x}} + \frac{\sqrt{y}}{2\sqrt{x}} \neq 0$$

Thus there are no extrema within the boundary. It therefore suffices to check that $f \leq 0$ on the boundary (f is then strictly < 0 inside since anywhere where $f=0$ would be an extrema).

For $y=1$ we get $-2 + 2\sqrt{x}$ which is true since $x < 1$
For $x=0$ we get $\sqrt{y} > 1/3$ which is true since $y > .25$
We need therefore only check $y = .25 + .75x$

Let $z = \sqrt{x}$ then we need show

$1 - 3\sqrt{.25 + .75z^2} + z + z\sqrt{.25 + .75z^2} \leq 0$

iff $1 + z \leq (3-z)\sqrt{.25 + .75z^2}$

iff $g(z) = 3z^4 - 18z^3 + 24z^2 - 14z + 5 \geq 0$

Now $g' = 12z^3 - 54z^2 + 48z - 14$

$g'' = 36z^2 - 108z + 48$ so $(g')'$ has one zero between 0 and 1, and one can check that g' is negative there.

Therefore $g' \leq 0$ for all z on the unit interval since both endpoints are negative which means that all possible extrema of g' are negative. Therefore g is decreasing. Also $g(1) = 0$ so $g \geq 0$.

QED

We may note that F can be 0. If $S^* = -.97$, $F^* = 1.97$ then prior S,F are $-.879$, $+.087$.

THEOREM B: $-2 < T < 1$

Proof: $T > -2$ iff $1 + 2\sqrt{m} - 3\sqrt{mm/m} + 2\sqrt{mm/m} - 2\sqrt{m} > 0$

iff $1 > \sqrt{mm/m}$, true.

$T < 1$ iff $1 + 3\sqrt{m} - 4\sqrt{mm/m} < 0$

iff $1 + 3\sqrt{x} - 4\sqrt{y} < 0$

$\frac{\partial f}{\partial x} = \frac{3}{2\sqrt{x}} \neq 0$

Along x=0 the theorem is true since $y \geq .25 > 1/16$
Along y=1 the theorem is true since $\sqrt{x} < 1$

On $y = .25 + .75x$ with $z = \sqrt{x}$ it suffices to show

$1 + 3z - 4\sqrt{.25 + .75z^2} < 0$ iff

$3z^2 - 6z + 3 > 0$ iff

$3(z-1)^2 > 0$, true

QED

T can be less than -1, e.g. $S^*=-.99$, $T^*=0$ give prior S and F of -.94, -.23, so T is < -1.

THEOREM C: $-1 < S < 2$

Proof: $S > -1$ iff $1 - \sqrt{mm/m} > 0$, true

$F > -1$ implies $-F < 1$

$T < 1$ implies $T-F < 2$, i.e. $S < 2$.

QED

S can exceed 1, e.g. $S^*=.99999952$, $F^*=-.33333349$ give priors of $S=1.147$, $F=-.668$.

People who have worked with series systems may be used to F always less than 0. As noted before, this may not always happen. But if the desired system prior has $F^* < 1$, such as in the uniform case, then F will be less than 0. This says that if the system is good enough to have failures < 1 then the components must work more often than they are tested.

THEOREM D: If additionally $F^* < 1$ then $F < 0$

Proof: It suffices to show $1 - 2\sqrt{mm/m} + \sqrt{mm} < 0$
i.e. $f(x,y) = 1 - 2\sqrt{y} + \sqrt{xy} < 0$

Now $F^* < 0$ implies $y > (1+x)/(3-x)$

$\frac{\partial f}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} = 0$ only if $y=0$ which is not in the region.

The boundary curves $x=0$, $y=1$ are trivial to evaluate.

On $y = (1+x)/(3-x)$ set $z = \sqrt{x}$

It suffices to show $1 - 2\sqrt{(1+z^2)/(3-z^2)} + z\sqrt{(1+z^2)/(3-z^2)} \leq 0$

iff $g(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 \geq 0$

$g' = 4z^3 - 12z^2 + 12z - 4$

$g'' = 12z^2 - 24z + 12 > 0$ so g' is increasing.

$g'(1) = 0$, thus $g' \leq 0$ so g is decreasing. $g(1) = 0$.

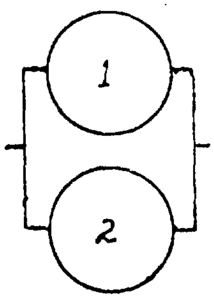
Therefore $g \geq 0$.

QED

These results may be summarized by the following:

THEOREM I: If S^*, F^* are independent series system prior parameters such that $-1 < S^*, F^* < 2$ and $-2 < T^* < 1$ then we may find component prior parameters S, F such that $-1 < F < 1$, $-1 < S < 2$ and $-2 < T < 1$ which give S^* and F^* for system parameters. If in addition $F^* < 1$ then $-1 < F < 0$.

INDEPENDENT PARALLEL SYSTEM



In this case,

$$m = 2\frac{S+1}{T+2} - \frac{(S+1)^2}{(T+2)^2}$$

$$mm = E((R_1 + R_2 - R_1R_2)^2)$$

$$= E(R_1^{\,2}) + 2E(R_1)E(R_2) + E(R_2^{\,2}) - 2E(R_1^{\,2})E(R_2)$$

$$-2E(R_1)E(R_2^{\,2}) + E(R_1^{\,2})E(R_2^{\,2})$$

$$= 2\frac{S+1}{T+2}\,\frac{S+2}{T+3} + 2\frac{(S+1)^2}{(T+2)^2} - 4\frac{(S+1)^2}{(T+2)^2}\frac{(S+2)}{(T+3)} + \frac{(S+1)^2}{(T+2)^2}\frac{(S+2)^2}{(T+3)^2}$$

Solving we get $\frac{S+1}{T+2} = 1 \pm \sqrt{1-m} = 1 - \sqrt{1-m}$

(since we wish $F > -1$)

Let $A = 1 - \sqrt{1-m}$, then $\frac{S+1}{T+2} = A$

$$2A\frac{S+2}{T+3} + 2A^2 - 4A^2\frac{(S+2)}{(T+3)} + A^2\frac{(S+2)^2}{(T+3)^2} = mm$$

So $$\frac{S+2}{T+3} = \frac{2A-1 \pm \sqrt{(1-2A)^2 - (2A^2-mm)}}{A}$$

$$= \frac{2A-1 + \sqrt{(1-2A)^2 - (2A^2-mm)}}{A}$$

$$= \frac{B}{A} \quad \text{where } B = 2A-1 + \sqrt{(1-2A)^2 - (2A^2-mm)}$$

(+ sign chosen since otherwise $B < A^2$ which would make $S < -1$)

We prove the following two weak inequalities:

(1) $B < A$

(2) $A^2 < B$

(1): We must show $A-1 + \sqrt{(2A-1)^2 - (2A^2-mm)} < 0$

iff $A^2 - 2A + mm < 0$

iff $1 - 2\sqrt{1-m} + 1 - m - 2 + 2\sqrt{1-m} + mm < 0$

iff $mm < m$, true since $F^* > -1$.

(2): We must show $2A - 1 - A^2 + \sqrt{(2A-1)^2 - 2A^2 + mm} > 0$

iff $A^4 - 4A^3 + 4A^2 - mm < 0$

iff $(2-m-2\sqrt{1-m})(2-m+2\sqrt{1-m}) < mm$

iff $(2-m)^2 - 4(1-m) < mm$

iff $m^2 < mm$

iff $\frac{S^*+1}{T^*+2} \frac{S^*+1}{T^*+2} < \frac{S^*+1}{T^*+2} \frac{S^*+2}{T^*+3}$

iff $\frac{S^*+1}{T^*+2} < \frac{S^*+2}{T^*+3}$

iff $F^* > -1$, true.

Now $\frac{S+1}{T+2} = A$ $\qquad \frac{S+2}{T+3} = \frac{B}{A}$

So solving for S and T we obtain

$S = \frac{-B - AB + 2A^2}{B-A^2}$ $\qquad T = \frac{A - 3B + 2A^2}{B-A^2}$

<u>THEOREM 1</u>: $-1 < S < 1$

Proof: $S > -1$ iff $A^2 - AB > 0$ iff $A > B$, True.

$S < 1$ iff $-2B - AB + 3A^2 < 0$

iff $A^4 + 10A^3 + (-20+mm)A^2 + 4mmA + 4mm > 0$

iff $8 - 18m - 8\sqrt{1-m} + 14m\sqrt{1-m} + m^2 + 10mm$

$-m \cdot mm - 6mm\sqrt{1-m} > 0$

iff $f(x,y) = 8 - 18x - 8\sqrt{1-x} + 14x\sqrt{1-x}$

$+ x^2 + 10xy - x^2y - 6xy\sqrt{1-x} > 0$

$\frac{\partial f}{\partial y} = 10x - x^2 - 6x\sqrt{1-x} = x(10-x-6\sqrt{1-x}) > 0$ except on

$x=0$. On $x=0$ we get $10y > 0$, true. On $y=1$ we get

$8 - 8x - 8\sqrt{1-x} + 8x\sqrt{1-x} > 0$ iff

$8(1-x) - 8(1-x)\sqrt{1-x} > 0$ iff

$1 - \sqrt{1-x} > 0$, true.

Thus it suffices to check f on $y = .25 + .75x$, i.e. we must show $8 - 18x - 8\sqrt{1-x} + 14x\sqrt{1-x} + x^2 + 10x(.25 + .75x)$
$-x^2(.25 + .75x) - 6x(.25 + .75x)\sqrt{1-x} \geq 0$

iff $3x^2 - 30x + 32 \geq 2\sqrt{1-x}(16-9x)$

iff $9x^4 + 144x^3 - 384x^2 + 256x \geq 0$

iff $g(x) = 9x^3 + 144x^2 - 384x + 256 \geq 0$

$g' = 27x^2 + 288x - 384$

$g'' = 54x + 288 > 0$ so g' is increasing, moreover

$g'(1) < 0$ so g' is < 0, hence g is decreasing with $g(1) > 0$

so $g \geq 0$.

QED

S can exceed 0, e.g. $S^* = 1.97$, $F^* = -.97$ give priors of $S = .088$, $F = -.879$.

<u>THEOREM 2</u>: $-2 < T < 1$

Proof: $T > -2$ iff $A - 3B + 2B - 2A^2 + 2A^2 > 0$

iff $A - B > 0$, true.

$T < 1$ iff $A - 4B + 3A^2 < 0$

iff $3A^2 - 7A + 4 < 4\sqrt{2A^2 - 4A + 1 - mm}$

iff $9A^4 - 42A^3 + 41A^2 + 8A - 16mm < 0$

iff $-6 + 13m + 6\sqrt{1-m} - 6m\sqrt{1-m} + 9m^2 - 16mm < 0$

iff $f(x,y) = -6 + 13x + 6\sqrt{1-x} - 6x\sqrt{1-x} + 9x^2 - 16xy < 0$

$$\frac{\partial f}{\partial y} = -16x = 0 \text{ only on } x = 0$$

f is trivially < 0 on $x = 0$. For $y = 1$ we get $(x-1)(9x+6) + 6\sqrt{1-x}(1-x) < 0$ iff $6\sqrt{1-x} < 9x + 6$, true since $x > 0$. Thus it suffices to look at $y = .25 + .75x$. We must show

$$-6 + 13x + 6\sqrt{1-x} - 6x\sqrt{1-x} + 9x^2 - 16x(.25 + .75x) \leq 0$$

iff $3(1-x)(x-2) + 6\sqrt{1-x}(1-x) \leq 0$

iff $3(x-2) + 6\sqrt{1-x} \leq 0$

iff $2\sqrt{1-x} < 2-x$

iff $4 - 4x < 4 - 4x + x^2$, true.

QED

<u>THEOREM 3</u>: $-1 < F < 2$

Proof: . $F = T-S = \dfrac{A - 2B + AB}{B-A^2} > -1$ iff

$A - B + AB - A^2 > 0$ iff $1 > A$, true.

$S > -1$ implies $-S < 1$

$T < 1$ implies $F = T-S < 2$

QED

F can exceed 1, e.g. $S^* = -.33333349$. $F^* = .99999952$

give priors of $S = -.668$, $F = 1.147$.

As in the series case, it is possible to gain more control if one makes an additional assumption.

<u>THEOREM 4</u>: If additionally $S^* < 1$ then $S < 0$

Proof: $S < 0$ iff $2A^2 < B + AB$

$$\text{iff} \quad (2A - 1 + \sqrt{(2A-1)^2 - 2A^2 + mm}\,)$$
$$+ A(2A - 1 + \sqrt{(2A-1)^2 - 2A^2 + mm}\,) > 2A^2$$

$$\text{iff} \quad 2A^4 + (-6 + mm)A^2 + 2mmA + mm > 0$$

$$\text{iff} \quad (2 - m - 2\sqrt{1-m}\,)(-2 - 2m + mm - 4\sqrt{1-m}\,)$$
$$+ mm(3 - 2\sqrt{1-m}\,) > 0$$

iff $\sqrt{1-m}(-4 + 8m - 4mm) + (4 - 10m + 2m^2 + 5mm - m \cdot mm) > 0$

iff $f(x,y) = \sqrt{1-x}(-4 + 8x - 4xy) + 4 - 10x + 2x^2$
$+ 5xy - x^2y > 0$

$\frac{\partial f}{\partial y} = -4x\sqrt{1-x} + 5x - x^2 = 0$ iff $x = 0$ or

$-4\sqrt{1-x} + 5 - x = 0$. If $-4\sqrt{1-x} + 5 - x = 0$ then

$-4 < -4\sqrt{1-x} = -5 + x < -4$ if $x < 1$, so this is

impossible. Thus the only possible extrema are on the boundary. f is trivially 0 on $x=0$. On $y=1$ we get

$\sqrt{1-x}(4x-4) + (4-5x+x^2) \geq 0$ iff

$4\sqrt{1-x} + x - 4 \leq 0$ iff

$x^2 + 8x \geq 0$, true. Thus it suffices to evaluate on the boundary curve which the hypothesis gives us. Now

$S^* < 1$ iff $y > \frac{3x}{2+x}$ so it suffices to evaluate on

the curve $y = \frac{3x}{2+x}$, that is, we must show

$\sqrt{1-x}(-4 + 8x - 4x(3x)/(2+x)) + 4 - 10x + 2x^2$

$+ 5x(3x)/(2+x) - x^2(3x)/(2+x) \geq 0$

Let $z = \sqrt{1-x}$, then it suffices to show

$$z\left(-4 + 8(1-z^2) - 4(1-z^2)\frac{3(1-z^2)}{3-z^2}\right) + 4 - 10(1-z^2) + 2(1-z^2)^2$$

$$+ 5(1-z^2)\frac{3(1-z^2)}{3-z^2} - (1-z^2)^2\frac{3(1-z^2)}{3-z^2} \geq 0$$

$$\text{iff} \quad (3-z^2)(2z^4 - 8z^3 + 6z^2 + 4z - 4) + (3-3z^2)(-z^4$$

$$+ 4z^3 - 3z^2 - 4z + 4) \geq 0$$

$$\text{iff} \quad z^6 - 4z^5 + 6z^4 - 4z^3 + z^2 \geq 0$$

$$\text{iff} \quad z^2(z-1)^4 \geq 0, \quad \text{true.}$$

QED

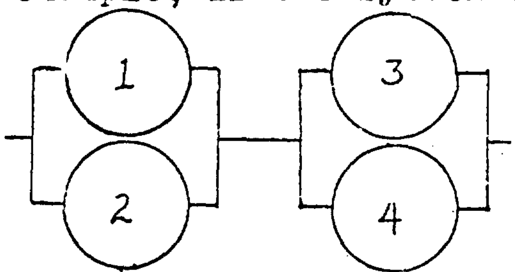
These results may be summarized by the following:

<u>THEOREM II</u>: If S^*,F^* are independent parallel system prior parameters such that $-1 < S^*,F^* < 2$ and $-2 < T^* < 1$ then we may find component prior parameters S,F such that $-1 < S < 1$, $-1 < F < 2$ and $-2 < T < 1$ which give S^* and F^* for system parameters. If in addition $S^* < 1$ then $-1 < S < 0$.

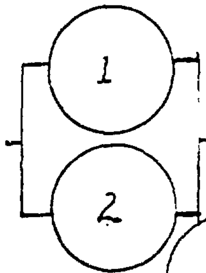
THE GENERAL CASE

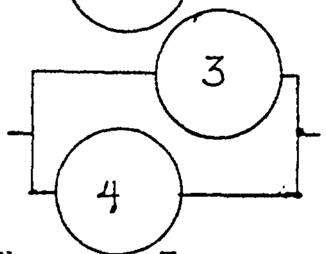
Consider now any system composed of independent components which can be broken down into series and parallel subsystems.[4] Suppose a system prior is desired with parameters S^*,F^* satisfying $-1 < S^*,F^* < 2$ and $-2 < T^* < 1$.

Then by repeated application of Theorems I and II one may derive priors for the components within the same limits. For example, if the system were



one could start with the subsystems A and B,

where A is 

and B is 

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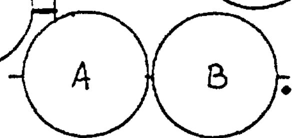
The system is then 

. By Theorem I we can find priors S,F within the proper range for A and B which give the desired system prior. Application of Theorem II to A and B individually yield priors for 1,2,3,4 which give A and B the subsystem priors of S,F, and thus give the system the desired prior. One has the following general theorem:

<u>THEOREM III</u>: If the desired system prior parameters S^*, F^* satisfy $-1 < S^*, F^* < 2$ and $-2 < T^* < 1$ then there exist a set of component prior parameters S,F which fall into the same limits and give the system the desired prior distribution.

Proof: Apply Theorems I and II repeatedly to the subsystems of the system.

FOOTNOTES

[1]Cole, P.V.Z., Mardo, J.G., Seible, G., and Stephenson, A.R., A Tri-Service Bayesian Approach to Nuclear Weapons Reliability Assessment, Picatinny Arsenal Technical Report 4359, April, 1972.

[2]Ibid., pp. 27-29

[3]Ibid., pp. 45-47

[4]Ibid., pp. 5-16